

SO/Sp Chern-Simons Gauge Theories At Large N , SO/Sp Penner Models And The Gauge Group Volumes

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Abstract

We construct a deformed SO/Sp Penner generating function responsible for the close connection between SO/Sp Chern-Simons gauge theories at large N and the SO/Sp Penner models. This construction is then shown to follow from a sector of a Chern-Simons gauge theory with coupling constant λ . The free energy and its continuum limit of the perturbative Chern-Simons gauge theory are obtained from the Penner model. Finally, asymptotic expansions for the logarithm of the gauge group volumes are given for every genus $g \geq 0$ and shown to be equivalent to the continuum limits of the SO/Sp Chern-Simons gauge theories and the SO/Sp Penner models.

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1 Introduction

The free energy of the Penner model [1, 2] is the generating function of the orbifold Euler characteristic of the moduli space of Riemann surfaces of genus g , with n punctures. The $SU(N)$ perturbative Chern-Simons free energy, based on the $1/N$ expansion introduced by 't Hooft [3], and the Penner free energy have a similar topological expansion. The perturbative Chern-Simons free energy [4] may be written as

$$F = \sum_{g=0, h=1} C_{g,h} N^{2-2g} \lambda^{2g-2+h}, \quad (1)$$

where λ is the 't Hooft coupling constant and h is the number of faces (boundaries) of the triangulated Riemann surfaces. In the Penner model h is identical to the number of punctures. The coefficient $C_{g,h}$ was shown by Witten [5] to be identical to the partition function of the A-model topological open string theory of genus g with h boundaries on a six-dimensional target space T^*S^3 . It has been shown by the first author [6] that the coefficient $C_{g,h}$ are related to the orbifold Euler characteristic of the moduli space of Riemann surfaces of genus g with $2h$ punctures. For the perturbative $SO(N)$ Chern-Simons gauge theory [7], the topological expansion when N is even has the form

$$F = \sum_{g=0, h=1} C_{g,2h} (N-1)^{2-2g} \lambda^{2g-2+2h} + \sum_{g=0, h=1} \tilde{C}_{g,2h+1} (N-1)^{1-2g} \lambda^{2g+2h}, \quad (2)$$

where the first term corresponds to half of the topological expansion of the $SU(N)$ Chern-Simons free energy, while the second term is nothing but the topological expansion of the non-orientable $SO(N)$ Chern-Simons free energy. We will show that the coefficients $\tilde{C}_{g,2h+1}$ are related to the orbifold Euler characteristic of the non-orientable Riemann surfaces of genus g with $2h+1$ punctures [8, 9]. Our goal in this work is to push further the connection between the $SU(N)$ Chern-Simons and the Penner model observed by the first author [6], when the gauge group is $SO(N)/Sp(N)$. It has been shown [6], that computations in the perturbative $SU(N)$ Chern-Simons may be carried out using the Penner model. Here, we construct a deformed $SO(N)$ Penner generating function and show that it gives rise to the perturbative $SO(N)$ Chern-Simons gauge theory. Such a construction turns out to contain both the $SO(N)$ Penner generating function [10, 11], and a generating function for the non-orientable orbifold Euler characteristic with coupling constant $t/2$. A check for such a construction is given by computing the free energy of the perturbative $SO(N)$ Chern-Simons theory on S^3 in terms of the string coupling constant g_s and the Kähler parameter t , this will be given in section 3. We provide a proof for our construction based on a simple observation that the perturbative $SO(N)$ Chern-Simons gauge theory splits into two sectors, one with coupling constant λ and the other sector with coupling constant $-\lambda$, this will be given in section 4. The perturbative free energy $F^{SO}(\lambda, N)$ associated with coupling constant λ is shown to generate the orbifold Euler characteristic of the moduli space of orientable and non-orientable Riemann surfaces of genus g with n punctures. Das and Gomez [12] reproduced the nonperturbative terms in the $SU(N)$ Chern-Simons theory using the continuum limit of the perturbative $SU(N)$ Chern-Simons theory. The nonperturbative terms are known to be related to the volume of the $SU(N)$ gauge group [13], these terms also have been obtained using the Penner model [6]. In section 5, of this paper, the continuum limit for the perturbative $SO(N)/Sp(N)$ Chern-Simons gauge theories are obtained and shown to be related to the asymptotic expansions of $\log(\text{vol}(SO(2N+1)))$ and $\log(\text{vol}(Sp(2N-1)))$ respectively, given in section 6. The asymptotic expansions of $\log(\text{vol}(G))$, for $G = SO(2N+1)$, $G = Sp(2N-1)$ are derived explicitly for all genera $g \geq 0$, this is shown to follow

simply from the asymptotic expansions of the Barnes and the Gamma functions [13, 14]. Also, the asymptotic expansion of $\log(\text{vol}(SO(2N)))$ and $\log(\text{vol}(Sp(2N)))$ were shown to reproduce the continuum limit of the $SO(2N)/Sp(2N)$ Penner models derived in [11]. Finally, in the last section our work is summarized, and the relations between the gauge group volumes are deduced as well as the relation between $\log \frac{\text{vol}(SO(2N))}{\text{vol}(Sp(2N))}$ and the generating function for the orbifold Euler characteristic associated with the non-orientable Riemann surfaces of genus g with n punctures.

2 The $SO(N)/Sp(N)$ Penner Models

Here, we will briefly review the $SO(N)/Sp(N)$ Penner models and its continuum limit studied in [11]. The free energy of the $SO(N)/Sp(N)$ Penner models are the generating functions of the orbifold Euler characteristic of the moduli space of real algebraic curves [9]. The explicit expression for the topological expansion of the $SO(N)/Sp(N)$ Penner free energy $F^{So/Sp}$ [10] in terms of the genus g and the punctures p is:

$$F^{So/Sp}(t, N) = \frac{1}{2} \sum_{\substack{g \geq 0, p > 0 \\ 2-2g-p < 0}} \chi^O(\mathfrak{M}_{g,p})(2N)^p(t)^{2g+p-2} \\ \mp \sum_{\substack{g \geq 0, p > 0 \\ 1-2g-p < 0}} \chi^{NO}(\mathfrak{M}_{g,p})(2N)^p(t)^{2g+p-1}, \quad (3)$$

where $\chi^O(\mathfrak{M}_{g,p})$ and $\chi^{NO}(\mathfrak{M}_{g,p})$ are the orbifold Euler characteristic of the moduli space of complex and real algebraic curves respectively, given by:

$$\chi^O(\mathfrak{M}_{g,p}) = (-1)^p \frac{(2g+p-3)!(2g-1)}{(2g)!p!} B_{2g}, \\ \chi^{NO}(\mathfrak{M}_{g,p}) = (-1)^p \frac{1}{2} \frac{(2g+p-2)!(2^{2g-1}-1)}{(2g)!p!} B_{2g}, \quad (4)$$

where B_{2g} are the Bernoulli numbers. The first term in the $SO(N)/Sp(N)$ Penner free energy $F^{So/Sp}$ corresponds to the orientable surfaces contributions, and is equal to half of the free energy of the ordinary Penner model with the size of the matrix doubled, while the second term corresponds to the non-orientable contributions. The partition function of $SO(N)/Sp(N)$ Penner models $Z(t, N)^{So/Sp} = e^{F(t, N)^{So/Sp}}$, may be written as [11]:

$$Z(t, N)^{So/Sp} = \left[\left(\frac{\sqrt{2\pi t}(et)^{-(t-1)}}{\Gamma(\frac{1}{t})} \right)^N \prod_{p=1}^{2N} (1+pt)^{\frac{1}{2}(2N-p)} \right] \prod_{p \text{ odd}}^{2N-1} (1+pt)^{\pm \frac{1}{2}}. \quad (5)$$

It is clear from this expression that $Z(t, N)^{So/Sp}$ is the product of two partition functions in which the first is ordinary Penner model [1, 15], given by the term between the square brackets, while the second is the partition function for the non-orientable contributions coming from $SO(N)/Sp(N)$ Penner models [11]. By making the natural scaling limit $t \rightarrow \frac{-t}{2N}$ in the free energy $F^{So/Sp}(t, N)$, the continuum limit of the $SO(N)/Sp(N)$ Penner models [11] may be obtained either using the Euler-Maclaurin formula or by summing over all punctures in the expression for the free energy given by equation (3). Then, one defines a new coupling constant $\mu = 2N(1-t)$, such that μ is

kept fixed as $N \rightarrow \infty$, and $t \rightarrow 1$ (the double scaling limit). Having done so, the expression for the free energy in the continuum limit becomes

$$F(\mu)^{SO/Sp} = \frac{1}{4}\mu^2 \log \mu \mp \frac{1}{4}\mu \log \mu - \frac{1}{24} \log \mu \pm \frac{1}{24\mu} + \frac{1}{2} \sum_{g \geq 2} \frac{1}{(2g-2)} \frac{B_{2g}}{2g} \mu^{2-2g} \pm \sum_{g \geq 2} \frac{(2^{2g-1}-1)}{(2g-1)} \frac{B_{2g}}{4g} \mu^{1-2g}, \quad (6)$$

where the coefficients of μ^{2-2g} and μ^{1-2g} are the topological orbifold Euler characteristic without punctures $\chi^O(\mathfrak{M}_{g,0})$ and $\chi^{NO}(\mathfrak{M}_{g,0})$ respectively. It is interesting to note that the SO/Sp Penner models and the Penner model share the same critical point $t = 1$ [11].

3 Perturbative $SO(N)$ Chern-Simons Theory From The Orthogonal Penner Model

In this section we propose a deformed $SO(N)$ Penner generating function that gives rise to the perturbative $SO(N)$ Chern-Simons gauge theory. This is motivated by the established connection between the ordinary Penner model and the $SU(N)$ Chern-Simons gauge theory given by the first author [6]. However, this time our proposed generating function will contain both the $SO(N)$ Penner model as well as a term that generates the non-orientable orbifold Euler characteristic with coupling constant $\frac{t}{2}$ ¹. To that end, let us consider a combination of the orthogonal Penner models defined as follows:

$$\mathcal{F}(t, N) = F^{SO}(t, N) - 2F^{SO/Non}(\frac{t}{2}, N), \quad (7)$$

where $F^{SO/Non}(\frac{t}{2}, N)$ corresponds to the generating function for the non-orientable orthogonal Penner model with coupling constant $\frac{t}{2}$. The above deformed orthogonal Penner model in the scaling $t \rightarrow \frac{t}{2N}$, may be rewritten as

$$\begin{aligned} \mathcal{F}(t, N) &= \frac{1}{2} \sum_{\substack{g \geq 0, p > 0 \\ 2-2g-p < 0}} \chi^O(\mathfrak{M}_{g,p}) (2N)^{2-2g} (t)^{2g+p-2} \\ &\quad - \sum_{\substack{g \geq 0, p > 0 \\ 1-2g-n < 0}} \chi^{NO}(\mathfrak{M}_{g,p}) (2N)^{1-2g} (t)^{2g+p-1} \\ &\quad + 2 \sum_{\substack{g \geq 0, p > 0 \\ 1-2g-p < 0}} \chi^{NO}(\mathfrak{M}_{g,p}) (2N)^{1-2g} (\frac{t}{2})^{2g+p-1}. \end{aligned} \quad (8)$$

Let us now consider a sum of two deformed orthogonal Penner models, one with coupling constant t , and the other with coupling constant $-t$, such that the topological expansion of the free energy in both cases is given by equation (8). If the coupling constant in the deformed orthogonal Penner model is set to be equal to $\lambda/2\pi n$, where λ is the Chern-Simons coupling constant and n is a positive integer, and let $\mathbf{F}(\lambda, N)$ be the total free energy for the two deformed $SO(N)$ Penner

¹We set the size of the matrix equal to $2N$ for both the orthogonal and the symplectic Penner models, while in [7] the size of the matrix is set to be $N + a$, where $a = -1$ for the orthogonal case and $a = -1$ for the symplectic case, where they become identical in the large N limit.

models summed over n , then by using equation (8) one has

$$\begin{aligned} \mathbf{F}(\lambda, N) &= \frac{1}{2} \sum_{\substack{g \geq 0, p > 0 \\ 2-2g-p < 0}} \sum_{n=1}^{\infty} \chi^O(\mathfrak{M}_{g,p})(2N)^{2-2g} (\lambda/2\pi n)^{2g+p-2} (1 + (-1)^p) \\ &+ 2 \sum_{\substack{g \geq 0, p > 0 \\ 1-2g-p < 0}} \sum_{n=1}^{\infty} \chi^{NO}(\mathfrak{M}_{g,p})(2N)^{1-2g} (\lambda/2\pi n)^{2g+p-1} \left(\frac{1}{2^{2g+p-1}} - \frac{1}{2} \right) (1 + (-1)^{p-1}). \end{aligned} \quad (9)$$

The first term on the right hand side of the above equation (9), contributes to the total free energy only when p is even, while the second term contributes only when p is odd. Therefore, the total free energy $\mathbf{F}(\lambda, N)$ becomes

$$\begin{aligned} \mathbf{F}(\lambda, N) &= \sum_{\substack{g \geq 0, p > 0 \\ 1-g-p < 0}} \sum_{n=1}^{\infty} \chi^O(\mathfrak{M}_{g,2p})(2N)^{2-2g} (\lambda/2\pi n)^{2g+2p-2} \\ &+ 4 \sum_{\substack{g \geq 0, p > 0 \\ g+p > 0}} \sum_{n=1}^{\infty} \chi^{NO}(\mathfrak{M}_{g,2p+1})(2N)^{1-2g} (\lambda/2\pi n)^{2g+2p} \left(\frac{1}{2^{2g+2p}} - \frac{1}{2} \right). \end{aligned} \quad (10)$$

This result may be compared with the work of Sinha and Vafa [7], in which the first and the second terms in equation (10) are nothing but the orientable and the non-orientable surfaces contributions to the free energy of the $SO(N)$ Chern-Simons gauge theory respectively. At the level of the partition function, the connection between the perturbative $SO(N)$ Chern-Simons and the $SO(N)$ Penner partition functions may be written as

$$Z_{CS}^{SO}(\lambda) = \prod_{n=1}^{\infty} Z_d^{SO}(\lambda) Z_d^{SO}(-\lambda), \quad (11)$$

where $Z_d^{SO}(\lambda)$ stands for the partition function of the deformed $SO(N)$ Penner model with coupling constant λ , given by

$$Z_d^{SO}(\lambda) = \left[\left(\frac{\sqrt{\lambda/n} (e\lambda/2\pi n)^{-\frac{2\pi n}{\lambda}}}{\Gamma(2\pi n/\lambda)} \right)^N \prod_{p=1}^{2N} (1 + p\lambda/2\pi n)^{\frac{1}{2}(2N-p)} \right] \prod_{p \text{ odd}}^{2N-1} \frac{(1 + p\lambda/2\pi n)^{\frac{1}{2}}}{(1 + p\lambda/4\pi n)}. \quad (12)$$

Now, using this partition function $Z_d^{SO}(\lambda)$, and making the natural scaling $\lambda \rightarrow \frac{\lambda}{2N}$ then one can show that²

$$Z_{CS}^{SO}(\lambda) = \prod_{n=1}^{\infty} \left[\prod_{p=1}^{2N} (1 - (p\lambda/4N\pi n)^2)^{\frac{1}{2}(2N-p)} \right] \prod_{p \text{ odd}}^{2N-1} \frac{(1 - (p\lambda/4N\pi n)^2)^{\frac{1}{2}}}{(1 - (p\lambda/8N\pi n)^2)}. \quad (13)$$

The first term between the square brackets corresponds to the partition function of the $SU(N)$ Chern-Simons gauge theory, while, the second term may be identified with the non-orientable

²The term $\frac{\sqrt{\lambda/n} (e\lambda/2\pi n)^{-\frac{2\pi n}{\lambda}}}{\Gamma(2\pi n/\lambda)}$ may be expanded as in [15, 10]. However this term will be canceled out upon multiplying $Z_d^{SO}(\lambda)$ by $Z_d^{SO}(-\lambda)$.

contribution to the partition function of the perturbative $SO(N)$ Chern-Simons gauge theory. Explicitly, the $g = 0$, free energy reads

$$\begin{aligned}\mathbf{F}^0(\lambda, N) &= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{-1}{2p(2p-1)(2p-2)} (2N)^2 \left(\frac{\lambda}{2\pi n}\right)^{2p-2} \\ &+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{1}{(2p+1)(2p)} (2N) \left(\frac{\lambda}{2\pi n}\right)^{2p} \left(\frac{1}{2^{2p}} - \frac{1}{2}\right).\end{aligned}\tag{14}$$

Similarly for genus $g \geq 1$ one has³

$$\begin{aligned}\mathbf{F}(\lambda, N) &= \sum_{n=1}^{\infty} \sum_{\substack{g \geq 1, p > 0 \\ 1-g-p < 0}} \frac{(2g+2p-3)!(2g-1)}{(2g)!(2p)!} B_{2g}(2N)^{2-2g} \left(\frac{\lambda}{2\pi n}\right)^{2g+2p-2} \\ &- 2 \sum_{n=1}^{\infty} \sum_{\substack{g \geq 0, p > 0 \\ g+p > 0}} \frac{(2g+2p-1)!(2^{2g-1}-1)}{(2g)!(2p+1)!} B_{2g}(2N)^{1-2g} \left(\frac{\lambda}{2\pi n}\right)^{2g+2p} \left(\frac{1}{2^{2g+2p}} - \frac{1}{2}\right).\end{aligned}\tag{15}$$

Therefore, using the deformed $SO(N)$ Penner model described by equation (7), we succeeded in obtaining the perturbative $SO(N)$ Chern-Simons theory given by equation (10). The latter was obtained originally using the large N expansion for the partition function of the Chern-Simons theory for $SO(N)$ and $Sp(N)$ gauge groups [7]. Now the computation in the Chern-Simons gauge theory may be carried out using the deformed orthogonal Penner model and this will be done in the next subsections.

3.1 $SO(N)$ Chern-Simons Gauge Theory At Large N From The $SO(N)$ Penner Model

In this section we are going to use the deformed orthogonal Penner generating function to compute the worldsheet RP^2 contribution, as well as the higher genus contribution to the sum over all punctures of the free energy of the $SO(N)$ Chern-Simons gauge theory [7]. This serves as a mere check of our proposal. As the deformed $SO(N)$ Penner model contains half of the ordinary Penner model [11], and the fact that the $SO(N)$ Chern-Simons theory already contains half the $SU(N)$ Chern-Simons theory [7]. Then, we will concentrate only on the non-orientable contributions, since computations for the orientable contributions may be found in [6].

3.1.1 The Genus $g = 0$ Computation

The non-orientable contribution for the free energy from equation (8), when $g = 0$ reads

$$\mathcal{F}_0^{NO}(t, N) = -\frac{N}{2} \sum_{p=2}^{\infty} \frac{1}{p(p-1)} (-t)^{p-1} + N \sum_{p=2}^{\infty} \frac{1}{p(p-1)} \left(\frac{-t}{2}\right)^{p-1},\tag{16}$$

³To see this we make the following shift $p \rightarrow p-1$ in equation (10).

in obtaining the above equation, we have used the expression for $\chi^{NO}(\mathfrak{M}_{0,p})$ given by equation (4). The sum over boundaries (punctures) may be carried out using the identity

$$\sum_{p=2}^{\infty} \frac{1}{p(p-1)} (-t)^{p-1} = \left[1 - \left(\frac{1+t}{t} \right) \log(1+t) \right], \quad (17)$$

therefore, the genus zero contribution becomes

$$\mathcal{F}_0^{NO}(t, N) = \frac{2N}{t} \left[\frac{t}{4} - (1+t/2) \log(1+t/2) + \frac{1}{4}(1+t) \log(1+t) \right]. \quad (18)$$

Using the established relation between the perturbative $SO(N)$ Chern-Simons and the deformed $SO(N)$ Penner model summarized by equation (9), the sum over boundaries for the total free energy $\mathbf{F}_0^{NO}(\lambda, N)$, is

$$\begin{aligned} \mathbf{F}_0^{NO}(\lambda, N) = \frac{2N}{\lambda} \sum_{n=1}^{\infty} & \left[\frac{\lambda}{2} + 2\pi n(1 - \lambda/4\pi n) \log(1 - \lambda/4\pi n) - \frac{\pi n}{2}(1 - \lambda/2\pi n) \log(1 - \lambda/2\pi n) \right. \\ & \left. - 2\pi n(1 + \lambda/4\pi n) \log(1 + \lambda/4\pi n) + \frac{\pi n}{2}(1 + \lambda/2\pi n) \log(1 + \lambda/2\pi n) \right], \end{aligned} \quad (19)$$

or in a more compact form

$$\begin{aligned} \mathbf{F}_0^{NO}(\lambda, N) = \frac{2N}{\lambda} \sum_{n \in \mathbf{Z}, n \neq 0} & \left[\frac{\lambda}{4} + 2\pi n(1 - \lambda/4\pi n) \log(1 - \lambda/4\pi n) \right. \\ & \left. - \frac{\pi n}{2}(1 - \lambda/2\pi n) \log(1 - \lambda/2\pi n) \right]. \end{aligned} \quad (20)$$

The free energy may be written in terms of the worldsheet instanons $\exp(-t)$ as in [7], to do so we let $\mathbf{F}_0^{NO}(\lambda, N) = (\frac{2N}{\lambda}) \mathbf{F}_0^{NO}(\lambda)$ then differentiate $\mathbf{F}_0^{NO}(\lambda)$ with respect to λ , to give

$$\frac{d}{d\lambda} \mathbf{F}_0^{NO}(\lambda) = \sum_{n \in \mathbf{Z}, n \neq 0} \left[\frac{1}{4} \log(1 - \lambda/2\pi n) - \frac{1}{2} \log(1 - \lambda/4\pi n) \right]. \quad (21)$$

The sum in the above equation may be carried out using the following identity⁴

$$\sum_{n \in \mathbf{Z}, n \neq 0} \log(1 - \lambda/2\pi n) = \frac{i\lambda}{2} + \log(1 - e^{-i\lambda}) - \log \lambda - \frac{i\pi}{2}, \quad (22)$$

to obtain

$$\frac{d}{d\lambda} \mathbf{F}_0^{NO}(\lambda) = \left[\frac{1}{4} \log \frac{1 + e^{-i\lambda/2}}{1 - e^{-i\lambda/2}} + \frac{1}{4} \log \lambda + \frac{i\pi}{8} - \frac{1}{2} \log 2 \right]. \quad (23)$$

From the identity

$$\log \frac{1 + e^{-i\lambda/2}}{1 - e^{-i\lambda/2}} = 2 \sum_{n \text{ odd}} \frac{e^{-in\lambda/2}}{n}, \quad (24)$$

the differentiated free energy $\frac{d}{d\lambda} \mathbf{F}_0^{NO}(\lambda)$, reads

$$\frac{d}{d\lambda} \mathbf{F}_0^{NO}(\lambda) = \frac{1}{2} \sum_{n \text{ odd}} \frac{e^{-in\lambda/2}}{n} + \frac{1}{4} \log \lambda + \frac{i\pi}{8} - \frac{1}{2} \log 2. \quad (25)$$

⁴This identity is obtained from the product formula $\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2})$.

Now integrating $\frac{d}{d\xi} \mathbf{F}_0^{NO}(\xi)$ with respect to ξ from 0 to λ , and using the substitution $\lambda = -it$, where t is now is the Kähler parameter and replace $\frac{2N}{\lambda}$ by $\frac{i}{g_s}$ where g_s is the string coupling constant, then the total free energy $\mathbf{F}_0^{NO}(t, N)$, reads

$$\mathbf{F}_0^{NO}(t, N) \cong \frac{1}{g_s} \sum_{n \text{ odd}} \frac{e^{-nt/2}}{n^2}, \quad (26)$$

this expression is in complete agreement with that obtained by Sinha and Vafa [7], using computations in the perturbative $SO(N)$ Chern-Simons theory.

3.1.2 Higher Genus Computations

By following the same procedure used for the $g = 0$, the higher genus contributions to the free energy of $SO(N)$ Chern-Simons theory in terms of the worldsheet instantons may be obtained using the deformed $SO(N)$ Penner model. From equation (8), the non-orientable surfaces contributions to the deformed $SO(N)$ Penner model is

$$\begin{aligned} \mathcal{F}^{NO}(t, N) &= \sum_{\substack{g \geq 1, p > 0 \\ 1-2g-p < 0}} (-1)^p \frac{(2g+p-2)!(2^{2g-1}-1)}{(2g)! p!} B_{2g}(2N)^{1-2g} \left(\frac{t}{2}\right)^{2g+p-1} \\ &- \frac{1}{2} \sum_{\substack{g \geq 1, p > 0 \\ 1-2g-p < 0}} (-1)^p \frac{(2g+p-2)!(2^{2g-1}-1)}{(2g)! p!} B_{2g}(2N)^{1-2g} (t)^{2g+p-1}. \end{aligned} \quad (27)$$

From the identity

$$\frac{d^p}{dt^p} (1+t)^{1-2g} = (-1)^p \frac{(2g+p-2)!}{(2g-2)!} (1+t)^{1-2g-p}, \quad (28)$$

together with Maclaurin series expansion of the function $(1+t)^{1-2g}$, the summation over punctures in equation (27) reads

$$\mathcal{F}^{NO}(t, N) = \sum_{g \geq 1} \frac{2^{2g-1}-1}{4g(2g-1)} B_{2g} \left[t^{2g-1} - 2 \left(\frac{t}{2}\right)^{2g-1} + \left(\frac{2N(1+t)}{t}\right)^{1-2g} - 2 \left(\frac{2N(1+t/2)}{t/2}\right)^{1-2g} \right]. \quad (29)$$

Again using the established relation between the perturbative $SO(N)$ Chern-Simons and the deformed $SO(N)$ Penner model, the sum over boundaries for the total free energy is

$$\mathbf{F}^{NO}(\lambda, N) = \sum_{g \geq 1} \frac{2^{2g-1}-1}{4g(2g-1)} B_{2g} \left(\frac{\lambda}{2N}\right)^{2g-1} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left[2 \left(\frac{1}{4\pi n + \lambda}\right)^{2g-1} - \left(\frac{1}{2\pi n + \lambda}\right)^{2g-1} \right]. \quad (30)$$

The expression for the total free energy $\mathbf{F}^{NO}(\lambda, N)$ given by the above relation may be rewritten using the following relations⁵

$$\frac{d^{2g-1}}{d\lambda^{2g-1}} \log\left(1 + \frac{\lambda}{2\pi n}\right) = \frac{(2g-2)!}{(2\pi n + \lambda)^{2g-1}},$$

⁵Note that the terms that are not written in the approximation $\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \log(1 + \frac{\lambda}{2\pi n}) \cong \log(1 - e^{-i\lambda})$ would disappear upon differentiation $(2g-1)$ times.

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \log\left(1 + \frac{\lambda}{2\pi n}\right) \cong \log(1 - e^{-i\lambda}),$$

together with the identity given by equation (24), as follows

$$\mathbf{F}^{NO}(\lambda, N) \cong - \sum_{g \geq 1} \frac{2^{2g-1} - 1}{(2g)!} B_{2g} \left(\frac{\lambda}{2N}\right)^{2g-1} \frac{d^{2g-1}}{d\lambda^{2g-1}} \sum_{p \text{ odd}} \frac{\infty}{n} \frac{e^{-in\lambda/2}}{n}. \quad (31)$$

Carrying out the differentiation, we obtain

$$\mathbf{F}^{NO}(\lambda, N) \cong \sum_{g \geq 1} \frac{B_{2g}}{(2g)!} \left(\frac{\lambda}{2N}\right)^{2g-1} \left(1 - \frac{1}{2^{2g-1}}\right) (i)^{2g-1} \sum_{p \text{ odd}} \frac{\infty}{n} n^{2g-2} e^{-in\lambda/2}. \quad (32)$$

In terms of the string variable g_s , and Kähler parameter t , the above expression becomes

$$\mathbf{F}^{NO}(g_s, t) \cong \sum_{g \geq 1} \frac{B_{2g}}{(2g)!} g_s^{2g-1} \left(1 - \frac{1}{2^{2g-1}}\right) \sum_{p \text{ odd}} \frac{\infty}{n} n^{2g-2} e^{-nt/2}. \quad (33)$$

Equivalently, this may be rewritten in terms of the orbifold Euler characteristic without punctures $\chi^{NO}(\mathfrak{M}_{g,0})$, associated with moduli space of non-orientable surfaces as

$$\mathbf{F}^{NO}(g_s, t) \cong \sum_{g \geq 1} \frac{\chi^{NO}(\mathfrak{M}_{g,0})}{(2g-2)!} g_s^{2g-1} \sum_{p \text{ odd}} \left(\frac{n}{2}\right)^{2g-2} e^{-nt/2}. \quad (34)$$

As it is clear from this expression there are no constant maps - those maps for which the whole Riemann surface of genus g is mapped to a point. This can be understood from the deformed $SO(N)$ Penner free energy given by equation (29). Now, the coefficient of the first and the second terms will disappear when constructing the perturbative Chern-Simons free energy, since both of these terms are odd in t . This is unlike the $SU(N)$ case where the term related to the constant maps is even in t [6], and when constructing the perturbative $SU(N)$ Chern-Simons free energy their coefficient is identified with the Hodge integral. Now going back to equation (33), and taking into account contributions from $g = 0$ and using the following identity [16]

$$\text{csch}\left(\frac{ng_s}{2}\right) = \sum_{g=0} \frac{2B_{2g}}{(2g)!} (1 - 2^{2g-1}) \left(\frac{ng_s}{2}\right)^{2g-1},$$

Then the full free energy takes the form

$$\mathbf{F}^{NO}(g_s, t) \cong - \sum_{p \text{ odd}} \frac{e^{-nt/2}}{2n \sinh(ng_s/2)}. \quad (35)$$

This is in a complete agreement with [17] (see equation (3.2)). The restriction on the sum over n may be lifted to give

$$\mathbf{F}^{NO}(g_s, t) \cong -\frac{1}{2} \left[\sum_{n=1} \frac{e^{-nt/2}}{2n \sinh(ng_s/2)} - \sum_{n=1} (-1)^n \frac{e^{-nt/2}}{2n \sinh(ng_s/2)} \right]. \quad (36)$$

Now, we may extend our work to include the $Sp(N)$ case as well. To do so, we note that the $SO(N)/SP(N)$ free energies of the Penner models differ by a minus sign in front of the non-orientable part contribution this is the duality between $SO(N)/Sp(N)$ Penner models [10]⁶. Also

⁶More precisely $F^{SO}(t, N) = F^{Sp}(-t, -N)$.

this duality is present in the $SO(N)/Sp(N)$ perturbative Chern-Simons gauge theories. Therefore, it follows that defining a deformed $Sp(N)$ Penner model as in equation (7) will give rise to the perturbative $Sp(N)$ Chern-Simons. Hence,

$$\mathbf{F}_{Sp}^{NO}(g_s, t) \cong \frac{1}{2} \left[\sum_{n=1} \frac{e^{-nt/2}}{2n \sinh(n g_s/2)} - \sum_{n=1} (-1)^n \frac{e^{-nt/2}}{2n \sinh(n g_s/2)} \right], \quad (37)$$

this is exactly equivalent to set t equals to $t + 2\pi i$ in equation (36) as noted in [7], that is,

$$\mathbf{F}_{Sp}^{NO}(g_s, t) = \mathbf{F}_{SO}^{NO}(g_s, t + 2\pi i). \quad (38)$$

4 From The Perturbative $SO(N)$ Chern-Simons Gauge Theory To The $SO(N)$ Penner Model

In section 3, we obtained the perturbative $SO(N)/Sp(N)$ Chern-Simons gauge theories using the $SO(N)/Sp(N)$ Penner models by construction. Here, we will see that the former generates the orbifold Euler characteristic of the moduli space of orientable and non-orientable Riemann surfaces of genus g with n punctures, this in turns proves our construction. Therefore the Penner model may be thought of as a building block for the Chern-Simons gauge theory. The free energy of the perturbative $SO(N)/Sp(N)$ Chern-Simons gauge theories is given by the following term [7]

$$\begin{aligned} \mathbf{F}^{SO/Sp} &= \log Z(\lambda, N) \\ &= \sum_{j \geq 1}^{N-2} f(j) \left[\sum_{p=1}^{\infty} \log \left(1 - \frac{j^2 \lambda^2}{4\pi^2 p^2 (N+a)^2} \right) \right], \end{aligned} \quad (39)$$

for $SO(N)$ $a = -1$ and if N is even then the weight $f(j)$ is given by [7]

$$f(j) = \begin{cases} \frac{(N+1-j)}{2} & j \text{ odd } j < N/2, \\ \frac{(N-1-j)}{2} & j \text{ odd } j \geq N/2, \\ \frac{(N-j)}{2} & j \text{ even } j < N/2, \\ \frac{(N-j-2)}{2} & j \text{ even } j \geq N/2. \end{cases} \quad (40)$$

It is a simple observation that the free energy in equation (39) factorizes into two sectors one with coupling constant λ , and the other with coupling constant $-\lambda$ as

$$\mathbf{F}^{SO} = F^{SO}(\lambda, N) + F^{SO}(-\lambda, N). \quad (41)$$

Although, this is a simple observation it has a deep consequences, to see this let us consider the first part of the free energy in the above equation ⁷, that is,

$$\begin{aligned} F^{SO}(\lambda, N) &= \sum_{j \geq 1}^{N-2} f(j) \left[\sum_{p=1}^{\infty} \log \left(1 + \frac{j\lambda}{2\pi p(N-1)} \right) \right] \\ &= - \sum_{j \geq 1}^{N-2} f(j) \sum_{m=1}^{\infty} j^m \left(\frac{-\lambda}{2(N-1)\pi} \right)^m \frac{\zeta(m)}{m}, \end{aligned} \quad (42)$$

⁷The following computations are similar to those of Sinha and Vafa [7], see section four therein, however, here we will concentrate on one sector of the perturbative Cheren-simons free energy.

the sum over j gives

$$\sum_{j \geq 1}^{N-2} f(j)j^m = \sum_{j \geq 1}^{N-2} \frac{(N-1-j)}{2} j^m + \sum_{j \geq 1}^{N/2-1} j^m - 2^{m-1} \sum_{j \geq 1}^{N/2-1} j^m, \quad (43)$$

then the free energy $F^{SO}(\lambda, N)$ may be written explicitly as

$$\begin{aligned} F^{SO}(\lambda, N) &= -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{j=1}^{N-2} (N-1-j)j^m \left(\frac{-\lambda}{2\pi(N-1)} \right)^m \frac{\zeta(m)}{m} \\ &\quad - \sum_{m=1}^{\infty} \sum_{j=1}^{N/2-1} (1-2^{m-1})j^m \left(\frac{-\lambda}{2\pi(N-1)} \right)^m \frac{\zeta(m)}{m}. \end{aligned} \quad (44)$$

If we let

$$F^{SO}(\lambda, N) = F^O(\lambda, N) + F^{NO}(\lambda, N),$$

where

$$F^O(\lambda, N) = -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{j=1}^{N-2} (N-1-j)j^m \left(\frac{-\lambda}{2\pi(N-1)} \right)^m \frac{\zeta(m)}{m}. \quad (45)$$

Using the power sum formula

$$\sum_{j \geq 1}^{N-2} j^m = \frac{(N-1)^m}{m+1} - \frac{1}{2}(N-1)^m + \frac{1}{m+1} \sum_{g=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2g} B_{2g} (N-1)^{m+1-2g}, \quad (46)$$

one has

$$\begin{aligned} F^O(\lambda, N) &= -\frac{1}{2} \sum_{m=1}^{\infty} \frac{(N-1)^2}{m(m+1)(m+2)} \left(\frac{-\lambda}{2\pi} \right)^m \zeta(m) \\ &\quad - \frac{1}{2} \sum_{m=1}^{\infty} \sum_{g=1}^{\lfloor m/2 \rfloor} \frac{B_{2g}}{(2g)!} \frac{(m-1)!}{(m+2-2g)} (1-2g) \left(\frac{-\lambda}{2\pi(N-1)} \right)^m \zeta(m), \end{aligned} \quad (47)$$

and letting $m = 2g - 2 + n$, we obtain

$$F^O(\lambda, N) = \frac{1}{2} \sum_{\substack{g \geq 1, n > 0 \\ 1-g-n < 0}} \frac{(2g+n-3)!(2g-1)}{(2g)!(n)!} B_{2g} (N-1)^{2-2g} \left(\frac{-\lambda}{2\pi} \right)^{2g+n-2} \zeta(2g-2+n), \quad (48)$$

this shows that

$$F^O(\lambda, N) = \frac{1}{2} \sum_{\substack{g \geq 0, n > 0 \\ 2-2g-n < 0}} \chi^O(\mathfrak{M}_{g,n}) (N-1)^{2-2g} \left(\frac{\lambda}{2\pi} \right)^{2g+n-2} \zeta(2g-2+n), \quad (49)$$

that is, $F^{SO}(\lambda, N)$, is the generating function for the orbifold Euler characteristic $\chi^O(\mathfrak{M}_{g,n})$ given by equation (4). We now move to compute the last term in equation (44), namely,

$$F^{NO}(\lambda, N) = \sum_{m=1}^{\infty} \sum_{j=1}^{N/2-1} (1-2^{m-1})j^m \left(\frac{-\lambda}{2\pi(N-1)} \right)^m \frac{\zeta(m)}{m}, \quad (50)$$

using the power sum formula

$$\sum_{j \geq 1}^{N/2-1} j^m = \frac{1}{m+1} \left(\frac{N-1}{2} \right)^m + \frac{1}{m+1} \sum_{g=1}^{\lfloor \frac{m}{2} \rfloor} (2^{1-2g} - 1) \binom{m+1}{2g} B_{2g} \left(\frac{N-1}{2} \right)^{m+1-2g}, \quad (51)$$

$F^{NO}(\lambda, N)$ takes the following form

$$\begin{aligned} F^{NO}(\lambda, N) &= -\frac{1}{2} \sum_{m=1}^{\infty} \frac{(N-1)}{m(m+1)} \left(\frac{-\lambda}{4\pi} \right)^m \zeta(m) + \frac{1}{4} \sum_{m=1}^{\infty} \frac{(N-1)}{m(m+1)} \left(\frac{-\lambda}{2\pi} \right)^m \zeta(m) \\ &+ \sum_{m=1}^{\infty} \sum_{g \geq 1}^{\lfloor m/2 \rfloor} (2^{2g-1} - 1) \left(\frac{-\lambda}{4\pi} \right)^m (N-1)^{1-2g} \frac{B_{2g}}{(2g)!} \frac{(m-1)!}{(m+1-2g)!} \zeta(m) \\ &- \frac{1}{2} \sum_{m=1}^{\infty} \sum_{g \geq 1}^{\lfloor m/2 \rfloor} (2^{2g-1} - 1) \left(\frac{-\lambda}{2\pi} \right)^m (N-1)^{1-2g} \frac{B_{2g}}{(2g)!} \frac{(m-1)!}{(m+1-2g)!} \zeta(m). \end{aligned} \quad (52)$$

Now let $m = 2g + n - 1$, to obtain

$$\begin{aligned} F^{NO}(\lambda, N) &= \sum_{\substack{n \geq 0, g \geq 0 \\ 1-2g-n \leq 0}}^{\infty} (2^{2g-1} - 1) \frac{(2g+n-2)!}{n!(2g)!} B_{2g} (N-1)^{1-2g} \left(\frac{-\lambda}{4\pi} \right)^{2g+n-1} \zeta(2g+n-1) \\ &- \frac{1}{2} \sum_{\substack{n \geq 0, g \geq 0 \\ 1-2g-n \leq 0}}^{\infty} (2^{2g-1} - 1) \frac{(2g+n-2)!}{n!(2g)!} B_{2g} (N-1)^{1-2g} \left(\frac{-\lambda}{2\pi} \right)^{2g+n-1} \zeta(2g+n-1). \end{aligned} \quad (53)$$

This equation may also be written in terms of the orbifold Euler characteristic of the moduli space of the non-orientable Riemann surfaces, then the free energy $F^{SO}(\lambda, N)$, becomes

$$\begin{aligned} F^{SO}(\lambda, N) &= \frac{1}{2} \sum_{\substack{g \geq 0, n > 0 \\ 2-2g-n < 0}} \chi^O(\mathfrak{M}_{g,n}) (N-1)^{2-2g} \left(\frac{\lambda}{2\pi} \right)^{2g+n-2} \zeta(2g+n-2) \\ &- \sum_{\substack{g \geq 0, n > 0 \\ 1-2g-n < 0}} \chi^{NO}(\mathfrak{M}_{g,n}) (N-1)^{1-2g} \left(\frac{\lambda}{2\pi} \right)^{2g+n-1} \zeta(2g+n-1) \\ &+ 2 \sum_{\substack{g \geq 0, n > 0 \\ 1-2g-n < 0}} \chi^{NO}(\mathfrak{M}_{g,n}) (N-1)^{1-2g} \left(\frac{\lambda}{4\pi} \right)^{2g+n-1} \zeta(2g+n-1). \end{aligned} \quad (54)$$

This shows that the perturbative free energy of Chern-Simons gauge theory with coupling constant λ , namely, $F^{SO}(\lambda, N)$, generates the virtual orbifold Euler characteristic of the moduli space of orientable and non-orientable Riemann surfaces for all genera $g \geq 0$ with n punctures, i.e, $\chi^O(\mathfrak{M}_{g,n})$ and $\chi^{NO}(\mathfrak{M}_{g,n})$ respectively. This also, proves our proposition given in section 3.

5 The Continuum Limit of the Perturbative $SO(N)/Sp(N)$ Chern-Simons Theory

Having identified the Perturbative $SO(N)$ Chern-Simons free energy with the deformed Orthogonal Penner model, we may use the latter to compute the continuum (double scaling) limit of the theory. In [6] it was noted that in order to obtain the continuum limit of the perturbative $SU(N)$ Chern-Simons gauge theory one has to sum over all boundaries which is equivalent to sum over all punctures in the Penner model, here, we will follow the same procedure to find the continuum limit of the $SO(N)$ Chern-Simons gauge theory. As the sum over all punctures of the deformed $SO(N)$ Penner free energy was obtained in section 3, the continuum limit of the $SO(N)$ Chern-Simons gauge theory is obtained by defining a new coupling constant⁸ ν_n given by $\nu_n = \frac{2\pi(2N)}{\lambda}(\frac{\lambda}{2\pi} - n)$ in the expressions for the free energy. Using equation (20) for $g = 0$, one has

$$\mathbf{F}_0^{NO}(\nu, N) \cong \sum_{n \in \mathbf{Z}^*} (-1)^{n+1} \left[\frac{\nu_n}{4} \log \nu_n + \frac{\nu_n}{4} \log \frac{\lambda}{4\pi n N} + \frac{i\pi \nu_n}{2} \right], \quad (55)$$

where \mathbf{Z}^* is the set of all positive and negative integers. Similarly for higher genus $g \geq 1$ equation (30), gives

$$\mathbf{F}_{g \geq 1}^{NO}(\lambda, N) = \sum_{g \geq 1} \sum_{n \in \mathbf{Z}^*} \chi^{NO}(\mathfrak{M}_{g,0}) (-1)^n \nu_n^{1-2g}. \quad (56)$$

The $SO(N)$ Chern-Simons coupling constant λ is related to the level of Kac-Moody algebra k , by $\lambda = \frac{2\pi(N-1)}{(k+N-2)}$, this shows that λ has a fundamental domain between 0 and 2π . Therefore, the natural critical double scaling limit would be

$$\lambda \rightarrow 2\pi \quad \nu_1 = \text{finite}.$$

Using this limit and keeping only the non analytic terms in the total free energy, one has

$$F^{NO}(\nu_1) = \frac{\nu_1}{4} \log \nu_1 - \sum_{g \geq 1} \chi^{NO}(\mathfrak{M}_{g,0}) \nu_1^{1-2g}. \quad (57)$$

Therefore, including the orientable contributions to the $SO(N)$ Chern-Simons gauge theory [6, 12], then the full continuum limit of the perturbative $SO(N)$ Chern-Simons reads

$$\begin{aligned} \mathbf{F}^{SO}(\nu_1) \cong & \frac{\nu_1^2}{4} \log \nu_1 - \frac{1}{24} \log \nu_1 + \frac{\nu_1}{4} \log \nu_1 - \frac{1}{24\nu_1} \\ & + \frac{1}{2} \left(\sum_{g \geq 2} \chi^O(\mathfrak{M}_{g,0}) \nu_1^{2-2g} - 2 \sum_{g \geq 2} \chi^{NO}(\mathfrak{M}_{g,0}) \nu_1^{1-2g} \right). \end{aligned} \quad (58)$$

It is interesting to note that this expression is equivalent to the continuum limit of the $SO(N)$ Penner model⁹ when ν_1 is replaced with $-\mu$ [11]. Simalrly, the continuum limit of the perturbative $Sp(N)$ Chern-Simons gauge theory is

$$\begin{aligned} \mathbf{F}^{Sp}(\nu_1) \cong & \frac{\nu_1^2}{4} \log \nu_1 - \frac{1}{24} \log \nu_1 - \frac{\nu_1}{4} \log \nu_1 + \frac{1}{24\nu_1} \\ & + \frac{1}{2} \left(\sum_{g \geq 2} \chi^O(\mathfrak{M}_{g,0}) \nu_1^{2-2g} + 2 \sum_{g \geq 2} \chi^{NO}(\mathfrak{M}_{g,0}) \nu_1^{1-2g} \right). \end{aligned} \quad (59)$$

⁸Here, we use the same coupling constant that gives the continuum limit of the $SU(N)$ Chern-Simons [12].

⁹This also appears in the $SU(N)$ Chern-Simons gauge theory[6].

In the next section we will compute $\log \text{vol}(G)$ for $G = SO(2N + 1)$ and $G = Sp(2N - 1)$ and show that the continuum limit of the perturbative $SO(N)/Sp(N)$ Chern-Simons gauge theories are reproduced.

6 The Gauge Group Volumes

Ooguri and Vafa [13] related $\log \text{vol}(G)$ for $G = SO(2N + 1)$, and $G = Sp(2N - 1)$ to the virtual Euler characteristic without punctures on the moduli space of orientable and non-orientable Riemann surfaces for genus $g \geq 2$. Here, we will give an alternative derivation for the asymptotic expansions of $\log(\text{vol}(SO(2N + 1)))$ and $\log(\text{vol}(Sp(2N - 1)))$ for all genera $g \geq 0$. It will be shown that the latter is related to the continuum limit of the perturbative $SO(N)/Sp(N)$ Chern-Simons gauge theories given in the last section. We will also show that the volume of the gauge groups $SO(2N)$, $Sp(2N)$ are equivalent to the continuum limit of the $SO(N)/Sp(N)$ Penner models [11]. The expression for $\text{vol}(SO(2N + 1))$ [13], is given by ¹⁰

$$\text{vol}(SO(2N + 1)) = \frac{2^{N+1}(2\pi)^{N^2+N-\frac{1}{4}}}{(2N-1)!(2N-3)! \cdots 3!1!}. \quad (60)$$

This may be written in terms of the Barnes function $G_2(z)$ [14, 13], defined by

$$G_2(N + 1) = \prod_{k=1}^{N-1} (N - k)!,$$

to give

$$\text{vol}(SO(2N + 1)) = \frac{2^{N+1}(2\pi)^{N^2+N-\frac{1}{4}}}{G_2(2N + 1)} \prod_{k=1}^{N-1} (2N - 2k)!. \quad (61)$$

Using the following identity

$$\log \prod_{k=1}^{N-1} (2N - 2k)! = \frac{1}{2} \log G_2(2N + 1) - \log(2N - 1)!!, \quad (62)$$

where $(2N - 1)!! = \frac{2^N}{\sqrt{\pi}} \Gamma(N + \frac{1}{2})$, then it is easy to show that

$$\log(\text{vol}(SO(2N + 1))) \cong -\frac{1}{2} \log G_2(2N + 1) - \frac{1}{2} \log \Gamma(N + \frac{1}{2}), \quad (63)$$

note that we have discarded terms which do not give rise to singularities. In the large N limit we may expand $\log \Gamma(N + \frac{1}{2})$, [19] as follows

$$\log \Gamma(N + \frac{1}{2}) \cong N \log N + \sum_{g=1}^{\infty} \frac{B_{2g}(\frac{1}{2})}{2g(2g-1)} (N)^{1-2g}, \quad (64)$$

where $B_{2g}(\frac{1}{2}) = (2^{1-2g} - 1)B_{2g}$, therefore, the above equation may be rewritten as

$$\log \Gamma(N + \frac{1}{2}) \cong N \log 2N - \sum_{g=1}^{\infty} \frac{B_{2g}(2^{2g-1} - 1)}{2g(2g-1)} (2N)^{1-2g}. \quad (65)$$

¹⁰This formula may also be found in the work of L. K. Hua in his classic text book [18].

On the other hand, the large N expansion for $\log G_2(2N+1)$ [13, 14], is

$$\log G_2(2N+1) \cong \frac{(2N)^2}{2} \log 2N - \frac{1}{12} \log 2N + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} (2N)^{2-2g}. \quad (66)$$

Finally, the large N expansion for $\log(\text{vol}(SO(2N+1)))$ to all genera reads

$$\begin{aligned} \log(\text{vol}(SO(2N+1))) &\cong -\frac{(2N)^2}{4} \log 2N + \frac{1}{24} \log 2N - \frac{(2N)}{4} \log 2N + \frac{1}{24} (2N)^{-1} \\ &- \sum_{g=2}^{\infty} \frac{B_{2g}}{4g(2g-2)} (2N)^{2-2g} + \sum_{g=2}^{\infty} \frac{B_{2g}(2^{2g-1}-1)}{4g(2g-1)} (2N)^{1-2g}, \end{aligned} \quad (67)$$

this expression may be written using the orbifold Euler characteristic as follows

$$\begin{aligned} -\log(\text{vol}(SO(2N+1))) &\cong \frac{(2N)^2}{4} \log 2N - \frac{1}{24} \log 2N + \frac{(2N)}{4} \log 2N - \frac{1}{24} (2N)^{-1} \\ &+ \frac{1}{2} \left(\sum_{g=2}^{\infty} \frac{\chi^O(\mathfrak{M}_{g,0})}{(2N)^{2g-2}} - 2 \sum_{g=2}^{\infty} \frac{\chi^{NO}(\mathfrak{M}_{g,0})}{(2N)^{2g-1}} \right). \end{aligned} \quad (68)$$

Similarly, the expression for $-\log(\text{vol}(Sp(2N-1)))$, is

$$\begin{aligned} -\log(\text{vol}(Sp(2N-1))) &\cong \frac{(2N)^2}{4} \log 2N - \frac{1}{24} \log(2N) - \frac{(2N)}{4} \log 2N + \frac{1}{24} (2N)^{-1} \\ &+ \frac{1}{2} \left(\sum_{g=2}^{\infty} \frac{\chi^O(\mathfrak{M}_{g,0})}{(2N)^{2g-2}} + 2 \sum_{g=2}^{\infty} \frac{\chi^{NO}(\mathfrak{M}_{g,0})}{(2N)^{2g-1}} \right). \end{aligned} \quad (69)$$

In obtaining the above equation we used the following relation [13],

$$\begin{aligned} \log(\text{vol}(SO(2N+1))) + \log(\text{vol}(Sp(2N-1))) &\cong -\frac{(2N)^2}{2} + \frac{1}{12} \log(2N) \\ &- \sum_{g=2}^{\infty} \frac{B_{2g}}{(2g)(2g-2)(2N)^{2g-2}}. \end{aligned} \quad (70)$$

Therefore, comparing the terms given by equations (68) and (69) with those of the continuum limits of the perturbative $SO(N)/Sp(N)$ Chern-Simons gauge theories given by equations (58) and (59) of the last section, we see that $-\log(\text{vol}(SO(2N+1)))$ is equivalent to the continuum limit of the perturbative $SO(N)$ Chern-Simons theory, when $2N$ is replaced by ν_1 . We also have the equivalence between $-\log(\text{vol}(Sp(2N-1)))$ and the continuum limit of the perturbative $Sp(N)$ Chern-Simons gauge theory. This shows that the coefficients in the continuum limit of SO/Sp Chern-Simons gauge theories are exactly those of the nonperturbative contributions given by equations (68) and (69), respectively, up to regular terms. Similar results for higher genus contributions to the $\log(\text{vol}(SO(2N+1)))$ may be found in [13], where the authors used different approach to obtain the expression for $\log(\text{vol}(SO(2N+1)))$. However, a factor of $+2$ should be inserted in front of the expression of the Euler characteristic of the moduli space of genus g with a single cross cap instead of minus one, see equation (4.31) in [13] and equation (65) above. Ogouri and Vafa [13], related the volume of $U(N)$ to the orbifold Euler characteristic $\chi(\mathfrak{M}_{g,0})$ of the moduli space of genus g Riemann surfaces as

$$-\log(\text{vol}(U(N))) \cong \frac{N^2}{2} \log N - \frac{1}{12} \log N + \sum_{g \geq 2} \frac{\chi(\mathfrak{M}_{g,0})}{N^{2g-2}}. \quad (71)$$

Note that, this equation is equivalent to the continuum limit of the Penner model [20, 15], provided one sets $N = \mu$. Here, it will be shown that the volume of the gauge groups $SO(2N)$, $Sp(2N)$ are equivalent to the continuum limit of the $SO(N)$ and $Sp(N)$ Penner models respectively [11], the volume of the gauge groups $SO(2N)$, $Sp(2N)$ are given by [13, 18]

$$\begin{aligned}\text{vol}(SO(2N)) &= \frac{\sqrt{2}(2\pi)^{N^2}}{(2N-3)!(2N-5)! \cdots 3!1!(N-1)!}, \\ \text{vol}(Sp(2N)) &= \frac{2^{-N}(2\pi)^{N^2+N}}{(2N-1)!(2N-3)! \cdots 3!1!},\end{aligned}\tag{72}$$

then it is not difficult to show that

$$\log(\text{vol}(SO(2N))) - \log(\text{vol}(Sp(2N))) \cong \log \Gamma(N + \frac{1}{2}),\tag{73}$$

and

$$\log(\text{vol}(SO(2N))) + \log(\text{vol}(Sp(2N))) \cong -\log G_2(2N+1),\tag{74}$$

as a result one has

$$\begin{aligned}-\log(\text{vol}(SO(2N))) &\cong \frac{1}{2} \log G_2(2N+1) - \frac{1}{2} \log \Gamma(N + 1/2) \\ &\cong \frac{1}{4}(2N)^2 \log 2N - \frac{1}{24} \log 2N - \frac{(2N)}{4} \log 2N + \frac{1}{24(2N)} \\ &+ \frac{1}{2} \sum_{g \geq 2} \left(\frac{\chi^O(\mathfrak{M}_{g,0})}{(2N)^{2g-2}} + 2 \frac{\chi^{NO}(\mathfrak{M}_{g,0})}{(2N)^{2g-1}} \right).\end{aligned}\tag{75}$$

Similarly,

$$\begin{aligned}-\log(\text{vol}(Sp(2N))) &\cong \frac{1}{2} \log G_2(2N+1) + \frac{1}{2} \log \Gamma(N + \frac{1}{2}) \\ &\cong \frac{1}{4}(2N)^2 \log 2N - \frac{1}{24} \log 2N + \frac{(2N)}{4} \log 2N - \frac{1}{24(2N)} \\ &+ \frac{1}{2} \sum_{g \geq 2} \left(\frac{\chi^O(\mathfrak{M}_{g,0})}{(2N)^{2g-2}} - 2 \frac{\chi^{NO}(\mathfrak{M}_{g,0})}{(2N)^{2g-1}} \right).\end{aligned}\tag{76}$$

If we let $2N = \mu$, then $-\log(\text{vol}(SO(2N)))$, $\log(\text{vol}(Sp(2N)))$ are the continuum limit of the $SO(2N)$ and $Sp(N)$ Penner models respectively [11], see equation (6) in this paper.

7 Discussion

In this paper we have related the SO/Sp Chern-Simons gauge theories at large N to the SO/Sp Penner models through deformed SO/Sp Penner generation functions by construction. This construction is then proved to be correct using a sector in the perturbative $SO(N)$ Chern-Simons free energy $F^{SO}(\lambda, N)$, with coupling constant λ , also it was shown that the latter generates the virtual orbifold Euler characteristic of the moduli space of orientable and non-orientable Riemann surfaces of genus g with n punctures. However, there is no restriction on the number of punctures unlike the $SO(N)$ Chern Simons gauge theory puts restrictions on the number of punctures. This

connection enables us to think of the perturbative $SO(N)/Sp(N)$ Chern-Simons gauge theory as two deformed $SO(N)$ Penner models of opposite coupling constants summed over all instantons. On the other hand, when summing over all punctures we end up with the perturbative $SO(N)$ Chern-Simons free energy written in terms of the orbifold Euler characteristic without punctures, since we used the free energy of Penner model which is known when summed over all punctures one obtains a free energy that contains the virtual Euler characteristic without punctures. Also in this paper we clarified the disappearance of the contribution from constant maps in the non-orientable part of the perturbative $SO(N)$ Chern-Simons free energy, through the Penner model. Both the free energy and the continuum limit in the SO/Sp Chern-Simons gauge theories were obtained using the SO/Sp Penner models. This is an extension to the work of Das and Gomez [12]. Also, it was shown that the asymptotic expansions of $\log(\text{vol}(G))$ where $G = SO(2N + 1)$, $G = Sp(2N - 1)$ are equivalent to the continuum limit of the perturbative SO/Sp Chern-Simons gauge theories respectively. However, when $G = SO(2N)$, $G = Sp(2N)$, the asymptotic expansions of the logarithm of the gauge groups are equivalent to continuum limit of the SO/Sp Penner models. Our computation of the asymptotic expansion for $\log(\text{vol}(SO(2N + 1)))$ is different from that of Ooguri and Vafa [13], and the result contains all genera. Comparing the two free energies of $SO(N)/Sp(N)$ Penner models and $SO(N)/Sp(N)$ Chern-Simons gauge theories it is clear that both have the same structure namely

$$F^{SO/Sp} = F^O \mp F^{NO}, \quad (77)$$

where the first term corresponds to half of the $SU(N)$ contribution, while, the second term is the non-orientable contribution. It was shown that the universal behavior of the topological partition function of the topological field theory (the Kodaira-Spencer theory) of the Calabi-Yau threefold near a conifold singularity was given by the free energy of the Penner model in the continuum limit [21]. Also it was shown that this topological partition function reduces to Chern-Simons theory on S^3 [22]. From this work, it is clear that the behavior of the continuum limit of both SO/SP Chern-Simons and SO/Sp Penner free energies are equivalent, Therefore, the topological partition function on the quotient of the resolved conifold by involution is given by the SO/Sp Penner free energy. This connection, however, needs both physical and mathematical explanation.

Next we will make some remarks on the relations between different gauge group volumes appeared in this paper, and the connection of these volumes with the generating function $\log \prod_{p \text{ odd}}^{2N-1} (1+pt)^{\frac{1}{2}}$ for $\chi^{NO}(\mathfrak{M}_{g,n})$ identified in [11]. From the formulae for $\text{vol}(SO(2N + 1))$ and $\text{vol}(Sp(2N))$, given by equations (60), and (72), it follows that their asymptotic expansions should be the same, see equations (68) and (76). Similarly $-\log(\text{vol}(Sp(2N - 1)))$ and $-\log(\text{vol}(SO(2N)))$, have the same asymptotic expansions see equations (69) and (75). This equivalence should follow from the volume of the gauge groups $G = Sp(2N - 1)$ and $G = SO(2N)$, to that end let $N \rightarrow N - \frac{1}{2}$, in the expression for $\text{vol}(Sp(2N))$, to give

$$\text{vol}(Sp(2N - 1)) \cong \prod_{k=1}^{N-1} \frac{1}{(2N - 2k)!}. \quad (78)$$

On the other hand, one can show that

$$\text{vol}(SO(2N)) \cong \prod_{k=1}^{N-1} \frac{(2N - 2k)!}{G_2(2N + 1)} \Gamma(N + \frac{1}{2}), \quad (79)$$

and from equation (62), it follows that

$$\left(\prod_{k=1}^{N-1} (2N - 2k)! \right)^2 \cong \frac{G_2(2N + 1)}{\Gamma(N + \frac{1}{2})}, \quad (80)$$

hence the equivalence of the $\text{vol}(SO(2N))$ and $\text{vol}(Sp(2N-1))$. From equation (73) it is seen that

$$\mp \frac{1}{2} \log \left(\frac{\text{vol}(SO(2N))}{\text{vol}(Sp(2N))} \right) \cong \mp \frac{(2N)}{4} \log(2N) \pm \sum_{g \geq 1} \chi^{NO}(\mathfrak{M}_{g,0}) \frac{1}{(2N)^{2g-1}}, \quad (81)$$

that is, $\mp \frac{1}{2} \log \left(\frac{\text{vol}(SO(2N))}{\text{vol}(Sp(2N))} \right)$ generates the non-orientable contributions to the free energy of the SO/Sp Penner models in the continuum limit. On the other hand, it is well known from Penner model that differentiating the free energy in the continuum limits n times with respect to the continuum variable μ brings back the punctures to the Riemann surface [20, 15]. Hence, differentiating equation (81) n times with respect to $2N$ will produce a generating function for the orbifold Euler characteristic of the moduli space of non-orientable Riemann surfaces of genus g with n punctures.

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